NON-STEADY FLOW OF A VISCO-PLASTIC MATERIAL BETWEEN PARALLEL WALLS

(NEUSTANOVIVSHEESIA TECHENIYE VIAZKO-PLASTICHNOGO MATERIALA MEZHDU PARALLEL'NYMI STENKAMI)

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A.I. SAFRONCHIK
(Saratov)

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The flow of certain materials such as peat muck, printing inks, bitumens, and cement and clay mortars are described well enough by equation (1.1).

At present, solutions have been obtained for many problems of steady visco-plastic flow [1,2,3]. Non-steady flow problems have not been investigated extensively; there are in the literature only approximate solutions of this type of problems [3,4]. This present paper presents an exact solution of a non-stationary problem for one-dimensional visco-plastic flow. The distribution of velocity and the law of change of the "core" of the flow is derived by the method of Kolodner [6]. By way of illustration, the flow is considered for steady drop in pressure.

1. A material is commonly called a visco-plastic material, if it follows Bingham's law, which for a one-dimensional flow has the form

$$\tau - \tau_0 = \pm \mu \frac{\partial v}{\partial n} \tag{1.1}$$

Here τ is the shear stress, τ_0 is the limiting shearing stress (limit of flow), μ is the coefficient of viscosity, n is normal to the direction of velocity; the sign of μ coincides with the sign of $\partial v/\partial n$.

We consider the flow of a visco-plastic material between two infinite parallel planes separated by a distance $2h$, under the action of a constant drop in pressure in the direction x. The system of coordinates is chosen so that the plane xz coincides with the plane of symmetry of the flow, and the y-axis is perpendicular to it. In this case the equation of motion will have the form

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 v_x}{\partial y} \tag{1.2}$$

$$v_y = v_z = 0, \qquad \frac{\partial v_x}{\partial x} = \frac{\partial v_x}{\partial z} = 0; \qquad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0 \tag{1.3}$$

From (1.2) and (1.3) it follows directly that $v_x = v_x(y, t)$ and $\partial p/\partial x = P(t)$. The material is assumed to be incompressible (ρ = const). The pressure drop is a given function of time, and for the determination of the only velocity component different from zero, namely v_x, we have equation (1.2). It is necessary to add to the latter the initial and boundary conditions for determination of v_x.

A visco-plastic material has the property that its flow begins only in regions where $\tau > \tau_0$; for $\tau \leqslant \tau_0$ the material behaves like an elastic body. The elastic region we will refer to hereafter as the "core" of the flow. It is evident that the maximum stress arises in the neighborhood of the wall, where the material behaves as a viscous fluid, and consequently satisfies the adhesion condition

$$v_x(h, t) = v_x(-h, t) = 0. \tag{1.4}$$

In the case of non-stationary flow the "core" is a function of time; it should be determined as a part of the solution. In order that the problem be correct, it is necessary to impose two conditions on the required "core". The first condition is obtained from the definition of the "core" itself. Thus at its boundaries $y = +y_0(t)$ we have $\tau = \tau_0$,

$$\partial v_x / \partial y = 0 \qquad \text{for} \qquad y = \pm y_0(t) \tag{1.5}$$

We have a second condition on the "core". The "core" may be considered as a body of variable mass, which changes with change in the cross-sectional area. Applying the law of the "conservation of momentum" in the differential form to the mass of the "core", having a volume of unit length and breadth, and a height $2y_0(t)$, we obtain

$$m\frac{dv_0}{dt} = F + (v_0 - v_1)\frac{dm}{dt} \tag{1.6}$$

where m is the mass of the "core", v_0 is its velocity, v_1 is the velocity of the particles separating (or attaching), and F are the surface forces.

In the case considered the particles are separating (or attaching) without impact, that is

$$v_0 = v_1, \qquad m\frac{dv_0}{dt} = F \tag{1.7}$$

or more particularly

$$\frac{dv_0}{dt} = -\frac{1}{\rho}\frac{\partial p}{dx} - \frac{\tau_0}{\rho y_0(t)} \tag{1.8}$$

Integrating (1.8) with respect to t, we obtain

$$v_0(t) = v_0(0) - \frac{1}{\rho}\int_0^t \left[\frac{\partial p}{\partial x} + \frac{\tau_0}{y_0(\sigma)}\right] d\sigma \tag{1.9}$$

The initial distribution of velocity is set down in the form

$$v_x(y, 0) = \begin{cases} F(y) & \text{for} \quad y_0(0) < y < h, \ -h < y < -y_0(0) \\ F[y_0(0)] & \text{for} \quad -y_0(0) \leqslant y \leqslant y_0(0) \end{cases} \tag{1.10}$$

Since the flow has a surface of symmetry, it is necessary to solve the problem only for a single region, for example $\{y_0(t) < y < h, \ t > 0\}$, the solution in the other region being obtained by change of the sign of the variable y and $y_0(t)$.

Introducing the dimensionless time $\xi = (\nu/h^2)t$, the dimensionless coordinate $\eta = y/h$ and the dimensionless velocity

$$u(\eta, \xi) = \frac{\mu l}{p_0 h} v_x(y, t)$$

where p_0/l is the characteristic drop in pressure per unit length, we reduce the equation and boundary equations to the dimensionless form

$$\frac{\partial u}{\partial \xi} = \frac{\partial^2 u}{\partial \eta^2} - P_*(\xi), \qquad P_*(\xi) = \frac{l}{p_0} P(t) \tag{1.11}$$

$$u(1, \xi) = 0, \qquad \frac{\partial u}{\partial \eta} = 0 \quad \text{for} \quad \eta = \delta(\xi) = \frac{y_0(t)}{h} \tag{1.12}$$

We shall transform the second condition of the "core" into

$$u_0(\xi) = u_0(0) - \int_0^{\xi} \left[P_*(\sigma) + \frac{S}{\delta(\sigma)} \right] d\sigma \qquad \left(S = \frac{\tau_0 l}{p_0 h} \right) \tag{1.13}$$

where S is a dimensionless parameter. For $S = 0$, the material goes over into a viscous fluid. The initial condition has the form

$$u(\eta, 0) = \frac{\mu l}{p h^2} F(y) = F_*^{*}(\eta) \qquad (\delta_0 \leqslant \eta \leqslant 1) \tag{1.14}$$

$$u_0(0) = F_*(\delta_0) \qquad (0 \leqslant \eta \leqslant \delta_0) \qquad \left(\delta_0 = \delta(0) = \frac{y_0(0)}{h} \right) \tag{1.15}$$

Here $y(0)$ is half the initial thickness of the "core".

2. For solution of the boundary problem "with the required boundary", we take advantage of the method of analytic extension [6,7], which permits deriving the equation for the "core" without knowledge of the velocity profile of the entire flow. We notice that the problem formulated above is more complicated version of the classical problem of Stefan.

Hereafter the boundary and initial conditions are understood as limiting, i.e.

$$\lim u(\eta, \xi) = 0 \quad \text{for } \eta \to 1 - 0 \quad (\xi > 0) \tag{2.1}$$

$$\lim u(\eta, \xi) = F_*(\delta_0) - \int_0^\xi \left[P_*(\sigma) + \frac{S}{\delta(\sigma)}\right] d\sigma \quad \text{for } \eta \to \delta(\xi) + 0 \quad (\xi > 0) \tag{2.2}$$

$$\lim \frac{\partial u}{\partial \eta} = 0 \quad \text{for } \eta \to \delta(\xi) + 0 \quad (\xi > 0) \tag{2.3}$$

$$\lim u(\eta, \xi) = F_*(\eta) \quad \text{for } \xi \to +0 \quad (\delta_0 < \eta < 1) \tag{2.4}$$

We shall seek a solution in the form

$$u(\eta, \xi) = w(\eta, \xi) + \lambda(\xi, \eta) - \int_0^\xi P_*(\sigma)\, d\sigma \tag{2.5}$$

where the function $w(\eta, \xi)$ and $\lambda(\xi, \eta)$ satisfy the equation

$$\frac{\partial w}{\partial \xi} = \frac{\partial^2 w}{\partial \eta^2}, \qquad \frac{\partial \lambda}{\partial \xi} = \frac{\partial^2 \lambda}{\partial \eta^2}$$

We require that the function $w(\eta, \xi)$ satisfy the condition

$$\lim w(\eta, \xi) = F_*(\eta) \quad \text{for } \xi \to +0 \tag{2.6}$$

Then the function $\lambda(\eta, \xi)$ will have a zero initial condition. For finding the function $w(\eta, \xi)$, it is necessary to inquire into the properties of $F_*(\eta)$. Since $F_*(\eta)$ gives the initial distribution of velocity, it may be considered continuous, and may be differentiated.

In the interval $(\delta_0 \leqslant \eta < 1)$, the function $F_*(\eta)$ changes from $F_*(1) = 0$ (the adhesion condition) to the maximum value $F_*(\delta_0)$ on the boundary of the "core"; the derivative $F_*'(\eta)$ is finite, since, within the accuracy of a constant factor, it gives the distribution of shear stress. We assume also that in the interval $(\delta_0 < \eta \leqslant 1)$ it satisfies the Lipschitz condition

$$|F_*(\eta) - F_*(\sigma)| < A|\eta - \sigma|$$

We consider the function

$$\Phi(\eta) = \begin{cases} F_*(\eta) & (\delta_0 < \eta \leqslant 1) \\ F_*(\delta_0) & (-\infty < \eta \leqslant \delta_0) \end{cases}$$

It is evident that $\Phi(\eta)$ is continuous, differentiable, and its derivative is finite and satisfies the Lipschitz condition in the interval $(-\infty < \eta \leqslant 1)$.

As the function $w(\eta, \xi)$ we take

$$w(\eta, \xi) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{1} \Phi(\sigma) \exp\left\{-\frac{(\eta - \sigma)^2}{4\xi}\right\} \frac{d\sigma}{\sqrt{\xi}} \tag{2.7}$$

This function is a regular solution of the equation of heat conduction. Making the substitution $\sigma = \eta + 2\sqrt{\xi}\beta$ under the integral sign, it is easy to show that for all internal points in the interval $(-\infty < \eta \leqslant 1)$

$$\lim_{\xi\to+0} w(\eta, \xi) = \frac{1}{\sqrt{\pi}} \lim_{\xi\to+0} \int_{-\infty}^{\eta^*} \Phi(\eta + 2\sqrt{\xi}\beta) e^{-\beta^2} d\beta = \Phi(\eta)$$

$$\left(\eta^* = \frac{1-\eta}{2\sqrt{\xi}}\right) \qquad (2.8)$$

At the point $M(1,0)$ the limit depends essentially on the path by which the limit is reached. For an approach to the limit along the line $\eta = 1$, the limit is equal to $1/2\,\Phi(1) = 0$.

For the function $\lambda(\eta, \xi)$ we shall have the following boundary problem:

$$\frac{\partial\lambda}{\partial\xi} = \frac{\partial^2\lambda}{\partial\eta^2} \qquad \begin{pmatrix}\delta(\xi) < \eta < 1 \\ 0 < \xi \leqslant \xi_0\end{pmatrix} \qquad (2.9)$$

$$\lim_{\eta\to1-0} \lambda(\eta, \xi) = \int_0^{\xi} P_*(\sigma)\, d\sigma - \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{1} \Phi(\sigma) \exp\left\{-\frac{[1-\sigma]^2}{4\xi}\right\} \frac{d\sigma}{\sqrt{\xi}} = f(\xi) \qquad (2.10)$$

$$\lim_{\eta\to\delta(\xi)+0} \lambda(\eta, \xi) = F_*(\delta_0) - S\int_0^{\xi} \frac{d\sigma}{\delta(\sigma)} - \qquad (2.11)$$

$$-\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{1} \Phi(\sigma) \exp\left\{-\frac{[\delta(\xi)-\sigma]^2}{4\xi}\right\} \frac{d\sigma}{\sqrt{\xi}} = \varphi(\xi)$$

$$\lim_{\eta\to\delta(\xi)+0} \frac{\partial\lambda}{\partial\eta} = \frac{1}{4\sqrt{\pi}} \int_{-\infty}^{1} \Phi(\sigma) \frac{\delta(\xi)-\sigma}{\xi^{3/2}} \exp\left\{-\frac{[\delta(\xi)-\sigma]^2}{4}\right\} d\sigma = \Psi(\xi) \qquad (2.12)$$

$$\lim_{\xi\to+0} \lambda(\eta, \xi) = 0 \qquad (2.13)$$

The function $f(\xi)$ is continuous and differentiable for all $\xi > 0$. The value of $f(0)$ is equal to zero. The derivative $f'(\xi)$ can have a finite number of discontinuities of the first order and has a singularity at $\xi = 0$. Let us establish the nature of this singularity (2.14)

$$|f'(\xi)| \leqslant |P_*(\xi)| + \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{1} |\Phi'(1 + 2\sqrt{\xi}\beta)|\, |\beta| \exp(-\beta^2)\, d\beta < A + \frac{B}{\sqrt{\xi}}$$

The function $\phi(\xi)$ is continuous and differentiable for all $\xi > 0$. For $\xi = 0$ it has a zero value. In clarifying the properties of the derivative $\phi'(\xi)$, it is necessary to assign certain limitations on the function $\delta(\xi)$. Hereafter we shall assume that $\delta(\xi)$ is a finite, continuously differentiable function which is nowhere equal to zero. Furthermore we require that

$$|\delta'(\xi)| \leqslant \frac{C}{\sqrt{\xi}} \tag{2.15}$$

The sign of the equation conforms to all the solutions known up to the present and obtained in explicit form.

Introducing a new variable of integration, and differentiating (2.11) with respect to ξ, we obtain

$$\varphi'(\xi) = -\frac{S}{\delta(\xi)} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\delta^*(\xi)} \Phi'(1 + 2\sqrt{\xi}\beta) \left[\delta'(\xi) + \frac{\beta}{\sqrt{\xi}}\right] \exp(-\beta^2)\, d\beta$$

$$\left((\delta^*(\xi) = \frac{1 - \delta(\xi)}{2\sqrt{\xi}} \right) \tag{2.16}$$

An estimate of the absolute value of the derivative yields

$$|\varphi'(\xi)| \leqslant a + \frac{b}{\sqrt{\xi}} \tag{2.17}$$

The function $\Psi(\xi)$ is continuous, finite, and satisfies the Lipschitz condition in the interval $(0 < \xi \leqslant \xi_0)$

$$|\Psi(\xi) - \Psi(\sigma)| < A|\xi - \sigma| \tag{2.18}$$

The properties of continuity and finiteness are evident; and the condition (2.18) comes from the differentiability of $\delta(\xi)$ and the Lipschitz condition for the function $\Phi'(\eta)$:

$$|\Psi(\xi) - \Psi(\sigma)| \leqslant \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\delta^*(\xi)} |\Phi'[\delta(\xi) + 2\sqrt{\xi}\beta] - \Phi'[\delta(\sigma) + 2\sqrt{\sigma}\beta]|\, e^{-\beta^2} d\beta +$$

$$+ \frac{1}{2\sqrt{\pi}} \int_{\delta^*(\xi)}^{\delta^*(\sigma)} |\Phi'(\delta(\sigma) + 2\sqrt{\sigma}\beta)|\, e^{-\beta^2} d\beta \leqslant A_1|\xi - \sigma| + A_2|\xi - \sigma| = A|\xi - \sigma|$$

It is now possible to construct a solution. We designate the region $\{\delta(\xi) < \eta < 1,\ 0 < \xi \leqslant \xi_0\}$ in which the solution is sought, by D_+. Let $D_-\{-\infty < \eta < \delta(\xi),\ 0 < \xi \leqslant \xi_0\}$ be a region supplementary to D_+. We extend the determination of the solution into the region D_-, letting $\lambda(\eta, \xi) \equiv 0$ for $Q(\eta, \xi)$ D_-. In such a determination, $\lambda(\eta, \xi)$ will satisfy all the imposed conditions. For the construction of such a function $\lambda(\eta, \xi)$, we consider the following problem.

Let $\bar{D}$ be the closure of the regions D_- and D_+ in the set

$$E\{-\infty < \eta \leqslant 1, 0 < \xi \leqslant \xi_0, |\eta - \delta_0| + |\xi| > 0\}$$

and D is the interior of $\bar{D}$. It is evident that neither D nor $\bar{D}$ depends on $\delta(\xi)$. We determine now

$$\lambda_{\pm}(\eta, \xi) = \lim \lambda(Q) \quad \text{for} \quad Q \to M(\eta, \xi) \quad (Q \in D_{\pm})$$

We derive the solution of (2.9) in the region D, satisfying conditions (2.10) and (2.13), and also two step conditions on the arbitrary curve $\eta = \delta(\xi)$

$$\lim_{\eta \to \delta(\xi)+0} \lambda(\eta, \xi) - \lim_{\eta \to \delta(\xi)- \to 0} \lambda(\eta, \xi) = \varphi(\xi), \quad \lim_{\eta \to \delta(\xi)+0} \frac{\partial \lambda}{\partial \eta} - \lim_{\eta \to \delta(\xi)-0} \frac{\partial \lambda}{\partial \eta} = \Psi(\xi) \quad (2.19)$$

We also require that

$$\lambda(-\infty, \xi) = 0, \qquad |\lambda_{\pm}(\eta, \xi)| < A, \qquad \left|\frac{\partial \lambda_{\pm}(\lambda, \xi)}{\partial \eta}\right| < B \qquad (2.20)$$

It will be shown below that such a problem has, moreover, a unique solution. We seek a solution in the form

$$\lambda(\eta, \xi) = \frac{1}{2\sqrt{\pi}} \int_0^{\xi} \frac{f(\sigma)(1-\eta)}{(\xi-\sigma)^{3/2}} \exp\left[-\frac{(1-\eta)^2}{4(\xi-\sigma)}\right] d\sigma +$$

$$+ \frac{1}{4\sqrt{\pi}} \int_0^{\xi} \frac{\varphi(\sigma)}{(\xi-\sigma)^{3/2}} \left\{[\eta - \delta(\sigma)] \exp\frac{-[\eta-\delta(\sigma)]^2}{4(\xi-\sigma)} - \right.$$

$$\left. - [2-\eta-\delta(\sigma)] \exp\frac{-[2-\eta-\delta(\sigma)]^2}{4(\xi-\sigma)}\right\} d\sigma - \frac{1}{2\sqrt{\pi}} \int_0^{\xi} \frac{\Psi(\sigma) + \varphi(\sigma)\delta'(\sigma)}{\sqrt{\xi-\sigma}} \times$$

$$\times \left\{\exp\frac{-[\eta-\delta(\sigma)]^2}{4(\xi-\sigma)} - \exp\frac{-[2-\eta-\delta(\sigma)]^2}{4(\xi-\sigma)}\right\} d\sigma \qquad (2.21)$$

for $\xi > \sigma$ and $\lambda(\eta, \xi) \equiv 0$ for $\xi \leqslant \sigma$. We show that (2.21) is the unique solution of the problem posed.

Formally $\lambda(\eta, \xi)$ satisfies equation (2.9), but in order that it shall be a solution, it is necessary that all the integrals entering into equation (2.21) shall be convergent. We designate by J_1, J_2, J_3 and J_4 the following expressions:

$$J_1 = \frac{1}{2\sqrt{\pi}} \int_0^{\xi} \frac{f(\sigma)(1-\eta)}{(\xi-\sigma)^{3/2}} \exp\frac{-(1-\eta)^2}{4(\xi-\sigma)} d\sigma \qquad (2.22)$$

$$J_2 = \frac{1}{\sqrt{\pi}} \int_0^{\xi} \varphi(\sigma) \left\{\frac{\eta-\delta(\sigma)}{4(\xi-\sigma)^{3/2}} - \frac{\delta'(\sigma)}{2(\xi-\sigma)^{1/2}}\right\} \exp\frac{-[\eta-\delta(\sigma)]^2}{4(\xi-\sigma)} d\sigma \qquad (2.23)$$

$$J_3 = -\frac{1}{2\sqrt{\pi}} \int_0^{\xi} \frac{\Psi(\sigma)}{\sqrt{\xi-\sigma}} \exp\frac{-[\eta-\delta(\sigma)]^2}{4(\xi-\sigma)} d\sigma \qquad (2.24)$$

$$J_4 = -\frac{1}{\sqrt{\pi}}\int_0^\xi \varphi(\sigma)\left\{\frac{2-\eta-\delta(\sigma)}{4(\xi-\sigma)^{3/2}} - \frac{\delta'(\sigma)}{2(\xi-\sigma)^{1/2}}\right\}\exp\frac{-[2-\eta-\delta(\sigma)]^2}{4(\xi-\sigma)}\,d\sigma + \\ + \frac{1}{2\sqrt{\pi}}\int_0^\xi \frac{\Psi(\sigma)}{\sqrt{\xi-\sigma}}\exp\frac{-[2-\eta-\delta(\sigma)]^2}{4(\xi-\sigma)}\,d\sigma \qquad (2.25)$$

The integral $J_1(\eta,\ \xi)$ is bounded in the region $(-\infty < \eta \leqslant 1,\ 0 \leqslant \xi \leqslant \xi_0)$. From the differentiability of $f(\xi)$ and the condition $f(0) = 0$, it follows that

$$J_1 = -\frac{2}{\sqrt{\pi}}\int_0^\xi f'(\sigma)\int_\infty^{\eta^*} e^{-\beta^2}d\beta d\sigma \quad \text{for } \eta < 1,\ \xi \geqslant 0 \quad \left(\eta^* = \frac{1-\eta}{2\sqrt{\xi-\sigma}}\right) \qquad (2.26)$$

For $\eta = 1$, $J_1 = 0$. From (2.26) it follows that

$$|J_1| \leqslant \frac{2}{\sqrt{\pi}}\int_0^\xi |f'(\sigma)|\left|\int_\infty^{\eta^*} e^{-\beta^2}d\beta\right| d\sigma \leqslant A\xi + B\sqrt{\xi} \qquad (2.27)$$

From this it follows that

$$\lim J_1 = 0 \quad \text{for } \xi \to +0, \qquad \lim J_1 = f(\xi) \quad \text{for } \eta \to 1-0,$$

From the continuity of J_1 follows the possibility of the limiting transformation under the integral sign

$$\lim_{\eta\to 1-0} J_1 = \frac{2}{\sqrt{\pi}}\int_0^\xi f'(\sigma)\left\{\lim_{\eta\to 1-0}\int_{\eta^*}^\infty e^{-\beta^2}d\beta\right\} d\sigma = f(\xi) \qquad (2.28)$$

Proceeding to the limit for $\eta \to -\infty$ under the integral sign in (2.26), we obtain

$$\lim J_1 = 0 \quad \text{for } \eta \to -\infty$$

The derivative $\partial J_1/\partial\eta$ is bounded in the region $(-\infty < \eta \leqslant 1,\ 0 \leqslant \xi \leqslant \xi_9)$.

Differentiating (2.26) with respect to η, we obtain

$$\frac{\partial J_1}{\partial\eta} = \frac{1}{\sqrt{\pi}}\int_0^\xi \frac{f'(\sigma)}{\sqrt{\xi-\sigma}}\exp\frac{-(1-\eta)^2}{4(\xi-\sigma)}\,d\sigma \qquad (2.29)$$

Therefore

$$\left|\frac{\partial J_1}{\partial\eta}\right| \leqslant \frac{1}{\sqrt{\pi}}\int_0^\xi \frac{A + B\sigma^{-1/2}}{\sqrt{\xi-\sigma}}\,d\sigma = \frac{2A}{\sqrt{\pi}}\sqrt{\xi} + B\sqrt{\pi} \qquad (2.30)$$

Integrating the expression (2.23) for J_2 by parts, and using the condition $\phi(0) = 0$, we get

$$J_2 = -\frac{1}{\sqrt{\pi}}\int_0^{\xi}\varphi'(\sigma)\int_{z(\eta,\xi)}^{z(\eta,\sigma)} e^{-\beta^2}d\beta d\sigma \tag{2.31}$$

where

$$z(\eta,\sigma) = \frac{\eta-\delta(\sigma)}{2\sqrt{\xi-\sigma}}, \qquad z(\eta,\xi) = \begin{cases} \infty & \eta > \delta(\xi) \\ 0 & \eta = \delta(\xi) \\ -\infty & \eta < \delta(\xi) \end{cases} \tag{2.32}$$

The integral $J_2(\eta,\xi)$ is bounded for $(0 \leqslant \xi \leqslant \xi_0)$ and arbitrary η

$$|J_2| \leqslant \frac{1}{\sqrt{\pi}}\int_0^{\xi}|\varphi'(\sigma)|\left|\int_{z(\eta,\xi)}^{z(\eta,\sigma)} e^{-\beta^2}d\beta\right| d\sigma \leqslant a\xi + b\sqrt{\xi} < A \tag{2.33}$$

From this it follows that

$$\lim_{\xi\to+0} J_2 = 0$$

$$\lim_{\eta\to-\infty} J_2 = \frac{1}{\sqrt{\pi}}\int_0^{\xi}\varphi'(\sigma)\left\{\lim_{\eta\to-\infty}\int_{-\infty}^{z(\eta,\xi)} e^{-\beta^2}d\beta\right\}d\sigma = 0$$

On the curve $\eta = \delta(\xi)$ the integral J_2 has a discontinuity, and

$$\lim_{\eta\to\delta(\xi)+0} J_2 - \lim_{\eta\to\delta(\xi)-0} J_2 = \varphi(\xi) \tag{2.34}$$

Indeed, computing the limit $J_2(\eta,\ \xi)$ for $\eta \to \delta(\xi) + 0$

$$\lim_{\eta\to\delta(\xi)+0} J_2 = \frac{1}{\sqrt{\pi}}\int_0^{\xi}\varphi'(\sigma)\int_{\vartheta}^{\infty} e^{-\beta^2}d\beta d\sigma \qquad \left(\vartheta = \frac{\delta(\xi)-\delta(\sigma)}{2\sqrt{\xi-\sigma}}\right) \tag{2.35}$$

and its value on the curve $\eta = \delta(\xi)$

$$J_2[\delta(\xi),\xi] = -\frac{1}{\sqrt{\pi}}\int_0^{\xi}\varphi'(\sigma)\int_0^{\vartheta} e^{-\beta^2}d\beta d\sigma \tag{2.36}$$

we obtain

$$\lim J_2 = J_2[\delta(\xi),\ \xi] + \tfrac{1}{2}\varphi(\xi) \qquad \text{for } \eta\to\delta(\xi)+0 \tag{2.37}$$

In an analogous manner the result may be obtained

$$\lim J_2 = J_2[\delta(\xi),\ \xi] - \tfrac{1}{2}\varphi(\xi) \qquad \text{for } \eta\to\delta(\xi)-0 \tag{2.38}$$

Subtracting (2.38) from (2.37), we obtain (2.34).

The derivative $\partial J_2/\partial\eta$ is bounded for $(0 < \xi \leqslant \xi_0)$ and arbitrary η. Differentiating (2.31) with respect to η, we have

$$\frac{\partial J_2}{\partial\eta} = -\frac{1}{2\sqrt{\pi}}\int_0^{\xi}\frac{\varphi'(\sigma)}{\sqrt{\xi-\sigma}}\exp\frac{-[\eta-\delta(\sigma)]^2}{4(\xi-\sigma)}d\sigma \tag{2.39}$$

Hence

$$\left|\frac{\partial J_2}{\partial \eta}\right| \leqslant \frac{1}{2\sqrt{\pi}} \int_0^{\xi} \frac{a + b\sigma^{-1/2}}{\sqrt{\xi - \sigma}}\, d\sigma = \frac{a}{\sqrt{\pi}} \sqrt{\xi} + b \frac{\sqrt{\pi}}{2} < A \tag{2.40}$$

The integral $J_3(\xi, \eta)$ is bounded in the region $(-\infty < \eta \leqslant 1,\ 0 \leqslant \xi \leqslant \xi_0)$

$$|J_3| \leqslant B \int_0^{\xi} \frac{d\sigma}{\sqrt{\xi - \sigma}} = 2B\sqrt{\xi} < A \tag{2.41}$$

From (2.41) follows $\lim\limits_{\xi \to +0} J_3 = 0$ for all η

$$\lim_{\eta \to -\infty} J_3 = -\frac{1}{2\sqrt{\pi}} \int_0^{\xi} \frac{\Psi(\sigma)}{\sqrt{\xi - \sigma}} \left\{ \lim_{\eta \to -\infty} \exp \frac{-[\eta - \delta(\sigma)]^2}{4(\xi - \sigma)} \right\} d\sigma = 0 \tag{2.42}$$

The propriety of the boundary transformation under the sign of the integral proceeds from the uniform convergence of $J_3(\eta, \xi)$.

The derivative $\partial J_3/\partial \eta$ is bounded for $0 < \xi \leqslant \xi_0$. For proof of this assertion, we shall write the derivative

$$\frac{\partial J_3}{\partial \eta} = \frac{1}{4\sqrt{\pi}} \int_0^{\xi} \frac{\Psi(\sigma)[\eta - \delta(\sigma)]}{(\xi - \sigma)^{3/2}} \exp \frac{-[\eta - \delta(\sigma)]^2}{4(\xi - \sigma)}\, d\sigma \tag{2.43}$$

in the form

$$\frac{\partial J_3}{\partial \eta} = \frac{\Psi(\xi)}{\sqrt{\pi}} \int_0^{\xi} \left\{ \frac{\eta - \delta(\sigma)}{4(\xi - \sigma)^{3/2}} - \frac{\delta'(\sigma)}{2(\xi - \sigma)^{1/2}} \right\} \exp \frac{-[\eta - \delta(\sigma)]^2}{4(\xi - \sigma)}\, d\sigma -$$

$$- \frac{1}{4\sqrt{\pi}} \int_0^{\xi} \frac{\Psi(\xi) - \Psi(\sigma)}{(\xi - \sigma)^{3/2}} [\eta - \delta(\sigma)] \exp \frac{-[\eta - \delta(\sigma)]^2}{4(\xi - \sigma)}\, d\sigma +$$

$$+ \frac{\Psi(\xi)}{2\sqrt{\pi}} \int_0^{\xi} \frac{\delta'(\xi)}{\sqrt{\xi - \sigma}} \exp \frac{-[\eta - \delta(\sigma)]^2}{4(\xi - \sigma)}\, d\sigma \tag{2.44}$$

We designate these expressions by K_1, K_2, and K_3 respectively. Evaluation of K_2 gives

$$|K_2| \leqslant \frac{1}{4\sqrt{\pi}} \int_0^{\xi} \frac{|\Psi(\xi) - \Psi(\sigma)|}{(\xi - \sigma)^{3/2}} |\eta - \delta(\sigma)| d\sigma < A\sqrt{\xi} < B \tag{2.45}$$

for all finite η and $(0 \leqslant \xi \leqslant \xi_0)$. For K_3 we have

$$|K_3| \leqslant \frac{|\Psi(\xi)|}{2\sqrt{\pi}} A \int_0^{\xi} \frac{d\sigma}{\sqrt{\sigma}\sqrt{\xi - \sigma}} < B \tag{2.46}$$

Change of variable under the integral sign gives for K_1 the expression

$$K_1 = \frac{\Psi'(\xi)}{\sqrt{\pi}} \int_{z(\eta,\,\xi)}^{\theta} e^{-\beta^2} d\beta \qquad \left(\theta = \frac{\eta - \delta_0}{2\sqrt{\xi}}\right) \tag{2.47}$$

where, as before $z(\eta, \xi)$ conforms with (2.32).

From (2.47) it follows that

$$|K_1| \leqslant |\Psi'(\xi)| < B \tag{2.48}$$

The derivative $\partial J_3/\partial\eta$ in passing the curve $\eta = \delta(\xi)$ undergoes a discontinuity

$$\lim_{\eta\to\delta(\xi)+0} \frac{\partial J_3}{\partial\eta} - \lim_{\eta\to\delta(\xi)-0} \frac{\partial J_3}{\partial\eta} = \Psi'(\xi) \tag{2.49}$$

This assertion follows from the continuity of K_2 and K_3 and from the character of the discontinuity of K_1 on the curve $\eta = \delta(\xi)$.

Repeating the necessary considerations, it is easy to establish that the integral $J_4(\eta, \xi)$ is a continuous function for all η and $0 \leqslant \xi \leqslant \xi_0$, for that integral $J_4 \to 0$ for $\xi \to +0$ and $-\infty < \eta \leqslant 1$, and $J_4 \to 0$ for $\eta \to -\infty$.

Further, the derivative $\partial J_4/\partial\eta$ is finite for $0 \leqslant \xi \leqslant \xi_0$; and continuous in the half-zone $(-\infty < \eta < 1,\ 0 < \xi \leqslant \xi_0)$.

From the properties of the integrals J_1, J_2, J_3 and J_4 enumerated above, it follows that $\lambda(\eta, \xi)$ in the form (2.21) satisfies equation (2.9) in the region D and the conditions for $\eta = 1$, for $\eta = -\infty$ and for $\xi = 0$. On the curve $\eta = \delta(\xi)$ the function $\lambda(\nu, \xi)$ and its derivative $\partial\lambda/\partial\eta$ have an assigned discontinuity. Furthermore, the function $\lambda(\eta, \xi)$ is finite, as is its derivative, in $\bar{D}$. This means that it is a solution to the problem that was posed. It remains to demonstrate the uniqueness of the solution obtained. We assume that two solutions exist, λ_1 and λ_2 having the enumerated properties. It is evident that their difference, $\lambda_0 = \lambda_1 - \lambda_2$, satisfies equation (2.9) within $\bar{D}$, on the boundaries it takes the value zero, is finite together with its derivative, in $\bar{D}$. Furthermore, λ_0 is continuous in D. Then, on the basis of a well-known theorem (see, for example [8], Chapter XXIX), $\lambda_0 \equiv 0$ in the region D, that is $\lambda_1 \equiv \lambda_2$. We return to the problem of interest. In (2.21) the arbitrary function $\delta(\xi)$ is introduced. If one requires that

$$\lim_{\eta\to\delta(\xi)-0} \lambda(\eta, \xi) = 0, \qquad \lim_{\eta\to\delta(\xi)-0} \frac{\partial\lambda}{\partial\eta} = 0 \tag{2.50}$$

conditions (2.11) and (2.12) will be fulfilled, and $\lambda(\eta, \xi)$ on the right of the curve $\eta = \delta(\xi)$ will yield the required solution.

The conditions (2.50) may be regarded as the equations for determination of the required boundaries of the "core".

Thus we have two equations for the determination of one unknown function $\delta(\xi)$. We show that any solution of the first equation (2.50) satisfies at the same time the second equation (2.50), and conversely.

We consider $\lambda(\eta, \xi)$ in the region D_-, that is, to the left of the curve $\eta = \delta(\xi)$. It satisfies the zero initial condition, becomes zero at $\eta \to -\infty$ and for $\eta \to \delta(\xi) - 0$, is finite, as is its derivative in the closed region D_-, and consequently $\lambda \equiv 0$ in the region D_-. Hence, it follows that $\partial\lambda/\partial\eta \equiv 0$ in the region D_-. Converse considerations are proved in an analogous manner. Writing equation (2.50) in detail, we have

$$\frac{2}{\sqrt{\pi}}\int_0^{\xi} f'(\sigma)\int_{\vartheta_1}^{\infty} e^{-\beta^2}d\beta\, d\sigma - \frac{1}{\sqrt{\pi}}\int_0^{\xi}\varphi'(\sigma)\left\{\int_0^{\vartheta} e^{-\beta^2}d\beta + \int_{\vartheta_2}^{\infty} e^{-\beta^2}d\beta\right\}d\sigma - \tag{2.51}$$

$$-\frac{1}{2\sqrt{\pi}}\int_0^{\xi}\frac{\Psi(\sigma)}{\sqrt{\xi-\sigma}}\left\{\exp\frac{-[\delta(\xi)-\delta(\sigma)]^2}{4(\xi-\sigma)} - \exp\frac{-[2-\delta(\xi)-\delta(\sigma)]^2}{4(\xi-\sigma)}\right\}d\sigma = \frac{\varphi(\xi)}{2}$$

$$\frac{1}{2\sqrt{\pi}}\int_0^{\xi}\frac{f'(\sigma)}{\sqrt{\xi-\sigma}}\exp\frac{-[1-\delta(\xi)]^2}{4(\xi-\sigma)}d\sigma - \frac{1}{2\sqrt{\pi}}\int_0^{\xi}\frac{\varphi'(\sigma)}{\sqrt{\xi-\sigma}}\times \tag{2.52}$$

$$\times\left\{\exp\frac{-[\delta(\xi)-\delta(\sigma)]^2}{4(\xi-\sigma)} - \exp\frac{-[2-\delta(\xi)-\delta(\sigma)]^2}{4(\xi-\sigma)}\right\}d\sigma +$$

$$+\frac{1}{4\sqrt{\pi}}\int_0^{\xi}\frac{\Psi(\sigma)}{(\xi-\sigma)^{3/2}}\left\{[\delta(\xi)-\delta(\sigma)]\exp\frac{-[\delta(\xi)-\delta(\sigma)]^2}{4(\xi-\sigma)} +\right.$$

$$\left.+[2-\delta(\xi)-\delta(\sigma)]\exp\frac{-[2-\delta(\xi)-\delta(\sigma)]^2}{4(\xi-\sigma)}\right\}d\sigma = \frac{\Psi(\xi)}{2}$$

$$\vartheta = \frac{\delta(\xi)-\delta(\sigma)}{2\sqrt{\xi-\sigma}}, \qquad \vartheta_1 = \frac{1-\delta(\xi)}{2\sqrt{\xi-\sigma}}, \qquad \vartheta_2 = \frac{2-\delta(\xi)-\delta(\sigma)}{2\sqrt{\xi-\sigma}}$$

for the condition $\delta(0) = \delta_0$.

If one of these equations should have a unique solution, the solution would give the required law of change of the "core" with time. The solution (2.51) or (2.52) along with (2.21) and (2.5) completely describe the flow.

3. We consider the flow for a constant pressure drop. Suppose at the instant $t = 0$, a pressure drom $-\partial p/\partial x = p/l$ is imposed on the visco-plastic medium at rest, this pressure drop being maintained constant for all subsequent times. It is evident that the flow begins only for $p > \tau_0 l/h$, since for $p \leqslant \tau_0 l/h$, the material would behave as an elastic body. Thus $0 \leqslant S < 1$, where $S = \tau_0 l/ph$ gives the ratio to the actual pressure of the pressure at which motion starts. For the functions $f(\xi)$, $\phi(\xi)$ and $\Psi(\xi)$ we have

$$f(\xi) = -\frac{p}{p_0}\xi, \qquad \varphi(\xi) = -S\int_0^{\xi}\frac{d\sigma}{\delta(\sigma)}, \qquad \Psi(\xi) = 0 \tag{3.1}$$

The function $\delta(\xi)$ is determined from equation (2.53), which in the case considered takes the form, for the condition $\delta(0) = 1$

$$\int_0^{\xi} \exp \frac{-[1-\delta(\xi)]^2}{4(\xi-\sigma)} \frac{d\sigma}{\sqrt{\xi-\sigma}} = \tag{3.2}$$

$$= \frac{S}{2}\int_0^{\xi} \left\{\exp \frac{-[\delta(\xi)-\delta(\sigma)]^2}{4(\xi-\sigma)} + \exp \frac{-[2-\delta(\xi)-\delta(\sigma)]^2}{4(\xi-\sigma)}\right\} \frac{d\sigma}{\delta(\sigma)\sqrt{\xi-\sigma}}$$

This nonlinear integral equation is of a type similar to the equation of Volterra. An exact solution is as yet difficult to obtain, so that we give an approximate solution for sufficiently small values of ξ. For small ξ in the right part of (3.2) it is possible to assume that $\delta(\xi) \approx \delta(0) = 1$; then

$$\int_0^{\xi} \exp \frac{-[1-\delta(\xi)]^2}{4(\xi-\sigma)} \frac{d\sigma}{\sqrt{\xi-\sigma}} = 2S\sqrt{\xi} \tag{3.3}$$

Using the substitution $\alpha(\sigma) = \dfrac{1-\delta(\xi)}{2\sqrt{\xi-\sigma}}$ and integrating by parts, we obtain

$$\exp\{-[\alpha(0)]^2\} - 2\alpha(0)\int_{\alpha(0)}^{\infty} e^{-\beta^2}d\beta = S \tag{3.4}$$

Investigation of this equation shows that it has a unique, and moreover, a positive solution.

For the boundary of the "core" and its velocity we obtain

$$\delta(\xi) = 1 - 2\alpha(0)\sqrt{\xi} \tag{3.5}$$

$$\lim_{\eta\to\delta(\xi)+0} \lambda = \frac{S}{\alpha(0)}\sqrt{\xi}\left\{1 + \frac{\ln[1-2\alpha(0)\sqrt{\xi}]}{2\alpha(0)\sqrt{\xi}}\right\} \tag{3.6}$$

where $\alpha(0)$ is the solution of (3.4). Formulas (3.5) to (3.6) are valid in the interval $(0 \leqslant \xi < 1/4\,[\alpha(0)]^2$. Reverting to the previous variable, we obtain

$$y_0(t) = h - 2\alpha(0)\sqrt{\nu t} \tag{3.7}$$

$$v_0(t) = \frac{p}{\rho l}\left\{t + \frac{\tau_0 l}{p\alpha(0)}\sqrt{\frac{t}{\nu}}\left[1 + \ln\frac{1-2\alpha(0)\sqrt{\nu t}/h}{2\alpha(0)\sqrt{\nu t}/h}\right]\right\} \tag{3.8}$$

The distribution of velocity in the flow can also be easily obtained, but it is not derived here.

We notice that the method considered above is useful only for determination of a solution, that is, for finite values of time.

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